TRINITY COLLEGE

**Semester One Examination, 2019**

**Question/Answer booklet**

# MATHEMATICS SPECIALIST UNIT 3

## Section Two: Calculator-assumed

# SOLUTIONS

Student number: In figures | | | | | | | | |

In words _____

Your name _____

## Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

## Materials required/recommended for this section

***To be provided by the supervisor***
This Question/Answer booklet
Formula sheet (retained from Section One)

***To be provided by the candidate***
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 8 | 8 | 50 | 54 | 35 |
| Section Two: Calculator-assumed | 13 | 13 | 100 | 98 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**See next page**

**Section Two: Calculator-assumed** **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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**Question 9** **(6 marks)**

Consider the following system of equations, where $a$ and $b$ are constants.

$$\begin{aligned} x - 2y + z &= 1 \\ 2x + 2y - z &= 5 \\ 2x + ay + 2z &= b \end{aligned}$$

For each of the following cases, determine the number of solutions that exist for the system and briefly interpret the system geometrically.

(a) $a = 2,\ b = -4.$ (3 marks)

| **Solution** |
|---|
| The system has $1$ solution.<br><br>The system represents three planes that intersect at the point $(2, -1, -3)$.<br>$\left\{\begin{matrix} x-2y+z=1 \\ 2x+2y-z=5 \\ 2x+a\times y+2z=b \end{matrix}\right\} \Big\vert_{x,y,z}$<br>$\left\{x=2, y=\frac{b-2}{a+4}, z=\frac{-(a-2\cdot b+8)}{a+4}\right\}$<br>ans $\vert$ a=2 $\vert$ b=−4<br>$\{x=2, y=-1, z=-3\}$ |
| **Specific behaviours** |
| ✓ number of solutions<br>✓ interpretation<br>✓ interpretation includes point of intersection |

(b) $a = -4,\ b = -2.$ (3 marks)

| **Solution** |
|---|
| The system has no solutions.<br><br>The system represents two parallel planes that are cut by a third non-parallel plane.<br>ans $\vert$ a=−4 $\vert$ b=−2<br>$\{x=2, \text{Undefined}, \text{Undefined}\}$ |
| **Specific behaviours** |
| ✓ number of solutions<br>✓ interpretation<br>✓ interpretation refers to parallel planes |

**See next page**

**Question 10** **(7 marks)**

(a) Determine the values of the real constant $p$ and the real constant $q$ given that $z + 3 - 2i$ is a factor of $z^3 + pz + q$.

(4 marks)

| **Solution** |
|---|
| Let $z = -3 + 2i$, then $z^3 = 9 + 46i$<br>Hence $9 + 46i - 3p + 2pi + q = 0$<br><br>Re parts: $9 - 3p + q = 0$<br>Im parts: $46 + 2p = 0$<br><br>Hence $p = -23, q = -78$ |
| **Specific behaviours** |
| ✓ identifies root and substitutes<br>✓ equates real and imaginary parts to zero<br>✓ solves for $p$<br>✓ correct values |

(b) Clearly show that $3 - i$ is a root of the equation $z^3 - 8z^2 + 22z - 20 = 0$. (2 marks)

| **Solution** |
|---|
| $z = 3 - i, 22z = 66 - 22i, 8z^2 = 64 - 48i, z^3 = 18 - 26i$<br><br>$z^3 - 8z^2 + 22z - 20 = 18 - 26i - 64 + 48i + 66 - 22i - 20$<br>$= 84 - 84 + 48i - 48i$<br>$= 0$ |
| **Specific behaviours** |
| ✓ shows expanded term for $z^3$<br>✓ fully expands all terms and sums to zero |

(c) State all three solutions of $z^3 - 8z^2 + 22z - 20 = 0$. (1 mark)

| **Solution** |
|---|
| $z = 2, 3 + i, 3 - i$ |
| **Specific behaviours** |
| ✓ correct solutions |

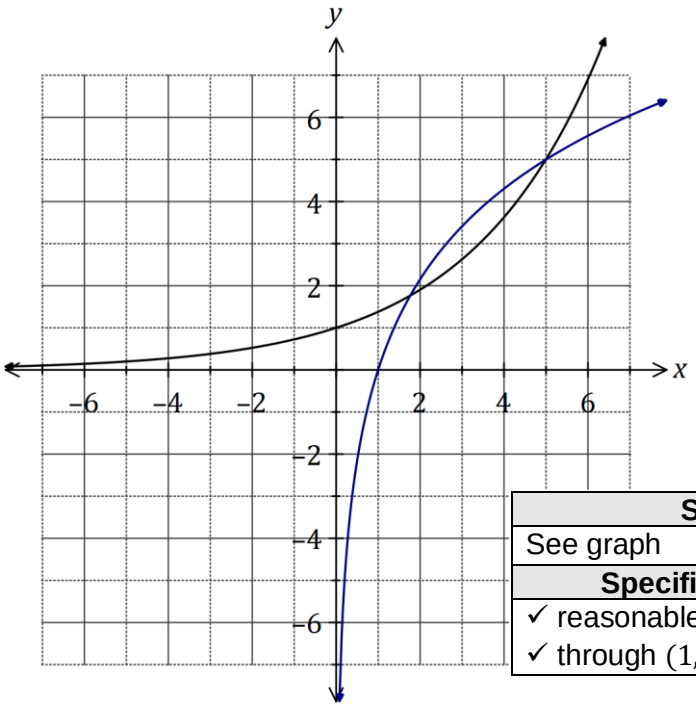
**See next page**

**Question 11** **(6 marks)**

(a) Explain why the function $f(x) = \sin x$, where $x \in \mathbb{R}$, is not one-to-one. (1 mark)

| Solution |
|---|
| Graph of $f(x)$ fails horizontal line test, etc |
| **Specific behaviours** |
| ✓ valid explanation |

(b) The graph of $y = g(x)$ is shown below. Sketch the graph of $y = g^{-1}(x)$ on the same axes. (2 marks)




| Solution |
|---|
| See graph |
| **Specific behaviours** |
| ✓ reasonable reflection in $y = x$ |
| ✓ through $(1, 0)$ and $(5, 5)$ |

(c) The inverse function of $h$ is defined as $h^{-1}(x) = x^2 + 10x + 22$ for $x \leq -5$. Determine the defining rule for $h(x)$ and state its domain. (3 marks)

| Solution |
|---|
| $x = (y+5)^2 - 3 \Rightarrow y = \pm\sqrt{x+3} - 5$ (CAS) |
| $D_{h^{-1}} = R_h \Rightarrow y \leq -5 \Rightarrow h(x) = -\sqrt{x+3} - 5$ |
| $D_h = \{x : x \in \mathbb{R}, x \geq -3\}$ |
| **Specific behaviours** |
| ✓ using CAS or otherwise obtains two possible functions |
| ✓ uses range of $h$ to determine $h(x)$ |
| ✓ states that $x \geq -3$ |

**See next page**

**Question 12** **(9 marks)**

The position vector of a small body is $\mathbf{r}(t) = (1 - 3\sin(t))\mathbf{i} + (2 + 2\cos(2t))\mathbf{j}$ where $t$ is the time in seconds since motion began.

(a) Show that the body is stationary when $t = \frac{\pi}{2}$ and state its position at this time. (3 marks)

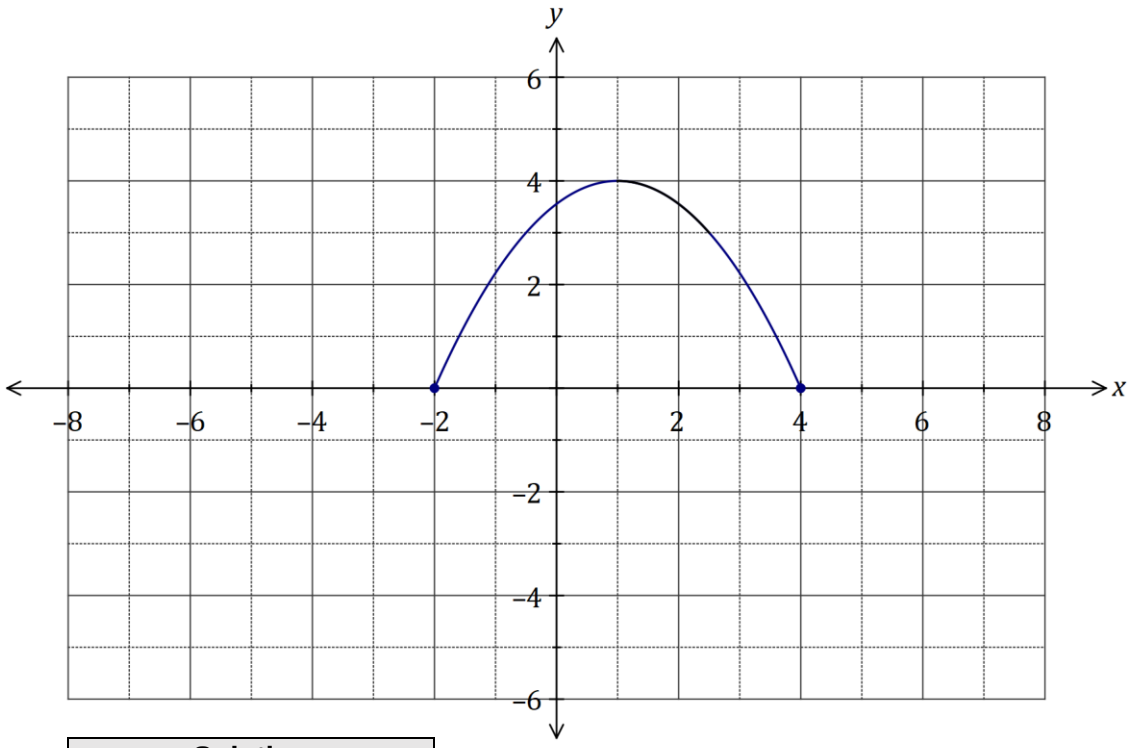
| **Solution** |
|---|
| $\mathbf{v}(t) = -3\cos(t)\,\mathbf{i} - 4\sin(2t)\mathbf{j}$<br><br>$\mathbf{v}\left(\frac{\pi}{2}\right) = -3\cos\left(\frac{\pi}{2}\right)\mathbf{i} - 4\sin(\pi)\,\mathbf{j} = \mathbf{0}$<br><br>$\mathbf{r}\left(\frac{\pi}{2}\right) = \left(1 - 3\sin\left(\frac{\pi}{2}\right)\right)\mathbf{i} + (2 + 2\cos(\pi))\mathbf{j} = -2\mathbf{i} + 0\mathbf{j}$ |
| **Specific behaviours** |
| ✓ expression for velocity<br>✓ substitutes time and obtains zero vector<br>✓ states position |

(b) Derive the Cartesian equation of the path of the body. (4 marks)

| **Solution** |
|---|
| $y = 2 + 2\cos(2t) = 2 + 2(1 - 2\sin^2(t)) = 4 - 4\sin^2(t)$<br><br>$x = 1 - 3\sin(t) \Rightarrow \sin^2(t) = \frac{(1-x)^2}{9}$<br><br>$y = 4 - \frac{4(1-x)^2}{9}$ where $-2 \leq x \leq 4$ |
| **Specific behaviours** |
| ✓ expression for $y$ in terms of $\sin^2(t)$<br>✓ expression for $\sin^2(t)$ in terms of $x$<br>✓ Cartesian equation<br>✓ restricts domain or range |

**See next page**

(c) Complete the following plot to show the path taken by the body. (2 marks)




| **Solution** |
|---|
| See graph |
| **Specific behaviours** |
| ✓ parabola |
| ✓ correct domain |

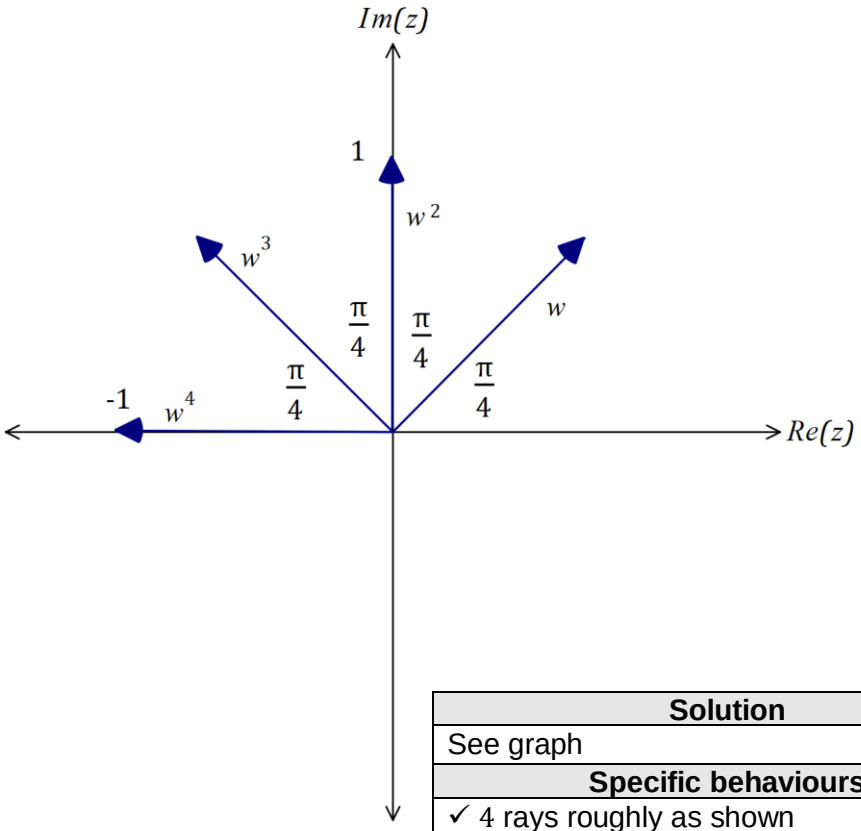
**Question 13** **(9 marks)**

Let $w = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

(a) Express $w, w^2, w^3$ and $w^4$ in the form $r\,\text{cis}\,\theta$, $-\pi < \theta \leq \pi$. (2 marks)

| Solution |
|---|
| $w = \text{cis}\left(\frac{\pi}{4}\right), w^2 = \text{cis}\left(\frac{\pi}{2}\right), w^3 = \text{cis}\left(\frac{3\pi}{4}\right), w^4 = \text{cis}(\pi)$ |
| **Specific behaviours** |
| ✓ $w$ correct |
| ✓ all correct |

(b) Sketch $w, w^2, w^3$ and $w^4$ as vectors on the Argand diagram below. (2 marks)




| Solution |
|---|
| See graph |
| **Specific behaviours** |
| ✓ 4 rays roughly as shown |
| ✓ indicates scale and angle between rays |

**See next page**

(c) Describe the transformation in the complex plane of any point $z$ when it is multiplied by $w$. (2 marks)

| **Solution** |
|---|
| Rotation about origin of $\frac{\pi}{4}$ anticlockwise |
| **Specific behaviours** |
| ✓ at least one element of transformation<br>✓ all three elements of transformation |

(d) Simplify

(i) $w + w^3 + w^5 + w^7$. (1 mark)

| **Solution** |
|---|
| 0 |
| **Specific behaviours** |
| ✓ correct value |

(ii) $w + w^3 + w^5 + \cdots + w^{2017} + w^{2019}$. (2 marks)

| **Solution** |
|---|
| $w + w^3 + w^5 + \cdots + w^{2015} = 0$<br><br>$w^{2017} + w^{2019} = w + w^3 = \sqrt{2}i$ |
| **Specific behaviours** |
| ✓ correct sum for $w + \cdots + w^{2015}$<br>✓ correct value |

**Question 14** **(8 marks)**

The position vectors of two particles at time $t$ are given below, where $a$ is a constant.

$$\mathbf{r}_A = 8\mathbf{i} - 5\mathbf{j} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \text{ and } \mathbf{r}_B = 3\mathbf{i} + a\mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

The paths of the particles cross at $P$ but the particles do not meet.

(a) Determine the value of the constant $a$ and the position vector of $P$. (5 marks)

**Solution**

$$\mathbf{r}_A = \begin{pmatrix} 8+t \\ -5+2t \\ -1-t \end{pmatrix}, \mathbf{r}_B = \begin{pmatrix} 3+3s \\ a-s \\ 1-2s \end{pmatrix}$$

Hence $8 + t = 3 + 3s$ and $-1 - t = 1 - 2s \Rightarrow t = 4, s = 3$

Using **j** coefficient: $-5 + 2(4) = a - 3 \Rightarrow a = 6$

$$\mathbf{r}_A(4) \Rightarrow \overrightarrow{OP} = \begin{pmatrix} 12 \\ 3 \\ -5 \end{pmatrix}$$

**Specific behaviours**

- ✓ replaces one $t$ with another variable (e.g. $s$)
- ✓ uses **i** and **k** components to write pair of equations
- ✓ solves equations for $t$ and $s$
- ✓ substitutes into **j** components and determines $a$
- ✓ uses $t$ or $s$ to find $P$

**ClassPad Solution**

Method 1

$$\left\{\begin{matrix} 8+t=3+3s \\ -5+2t=a-s \\ -1-t=1-2s \end{matrix}\right|_{t,s,a} \quad \{t=4, s=3, a=6\}$$

Method 2

$$\text{solve}\left(\begin{bmatrix} 8 \\ -5 \\ -1 \end{bmatrix} + t\times\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ a \\ 1 \end{bmatrix} + s\times\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}, \{t, s, a\}\right) \quad \{t=4, s=3, a=6\}$$

$$\begin{bmatrix} 8 \\ -5 \\ -1 \end{bmatrix} + t\times\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Big| t=4 \quad \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$$

$$\text{crossP}\left(\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right) \quad \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$$

$$\text{dotP}\left(\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ -5 \\ -1 \end{bmatrix}\right) \quad 28$$

(b) Show that the point $(1, -5, 4)$ lies in the plane containing the two lines. (3 marks)

**Solution**

$$(3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (5\mathbf{i} + \mathbf{j} + 7\mathbf{k})$$

$$(5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) \cdot (8\mathbf{i} - 5\mathbf{j} - \mathbf{k}) = 28$$

Equation of plane is $\mathbf{r} \cdot (5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) = 28$

$$(5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) \cdot (\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) = 5 - 5 + 28 = 28$$

Hence point lies in plane.

**Specific behaviours**

- ✓ calculates normal to plane
- ✓ calculates constant and writes equation of plane
- ✓ substitutes point, showing equation satisfied

**See next page**

**Question 15** **(7 marks)**

(a) Solve the equation $z^5 - 32i = 0$, writing your solutions in polar form $r\,\text{cis}\,\theta$. (4 marks)

| **Solution** |
|---|
| $z^5 = 32i$<br>$= 2^5\,\text{cis}\,\frac{\pi}{2}$<br><br>$z_n = 2\,\text{cis}\left(\frac{9\pi}{10} - \frac{4n\pi}{10}\right), n = 0,1,2,3,4$<br><br>$z_0 = 2\,\text{cis}\left(\frac{9\pi}{10}\right), z_1 = 2\,\text{cis}\left(\frac{\pi}{2}\right), z_0 = 2\,\text{cis}\left(\frac{\pi}{10}\right), z_0 = 2\,\text{cis}\left(\frac{-3\pi}{10}\right), z_0 = 2\,\text{cis}\left(\frac{-7\pi}{10}\right)$ |
| **Specific behaviours** |
| ✓ expresses in polar form<br>✓ states general solution<br>✓ states one correct solution in polar form<br>✓ states all correct solutions in polar form |

(b) Use your answers from (a) to show that $\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{3\pi}{10}\right) + \cos\left(\frac{7\pi}{10}\right) + \cos\left(\frac{9\pi}{10}\right) = 0$. (3 marks)

| **Solution** |
|---|
| Since $z_0 + z_1 + z_2 + z_3 + z_4 = 0$ then $\text{Re}(z_0 + z_1 + z_2 + z_3 + z_4) = 0$<br><br>$2\cos\left(\frac{9\pi}{10}\right) + 2\cos\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{10}\right) + 2\cos\left(-\frac{3\pi}{10}\right) + 2\cos\left(-\frac{7\pi}{10}\right) = 0$<br><br>But $\cos(-\theta) = \cos\theta$ and $\cos\frac{\pi}{2} = 0$<br><br>Hence $\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{3\pi}{10}\right) + \cos\left(\frac{7\pi}{10}\right) + \cos\left(\frac{9\pi}{10}\right) = 0$ |
| **Specific behaviours** |
| ✓ indicates that sum of roots is zero<br>✓ equates real part of sum of roots to zero<br>✓ states $\cos(-\theta) = \cos\theta$ and $\cos\frac{\pi}{2} = 0$ and simplifies |

**See next page**

**Question 16** **(9 marks)**

A pole and a wall stand vertically on horizontal ground. A small projectile is launched from the pole at a height of $4.25$ m above the ground and sometime later hits the wall at a height of $1.29$ m above the ground. The projectile has an initial velocity of $28$ ms$^{-1}$ at an angle of $54°$ above the horizontal.

Any effects of air resistance and wind can be ignored. Let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors in the horizontal and vertical (upward) directions and the foot of the pole be at $(0, 0)$.

The acceleration acting on the projectile is given by $\mathbf{a}(t) = -9.8\mathbf{j}$ ms$^{-2}$.

(a) Use the information above to derive vector equations for the velocity $\mathbf{v}(t)$ and displacement $\mathbf{r}(t)$ of the projectile at any time $t$. (3 marks)

| **Solution** |
|---|
| $\mathbf{v}(t) = (28\cos 54°)\mathbf{i} + (28\sin 54° - 9.8t)\mathbf{j}$<br>$\approx (16.458)\mathbf{i} + (22.652 - 9.8t)\mathbf{j}$<br><br>$\mathbf{r}(t) = (28t\cos 54°)\mathbf{i} + (4.25 + 28t\sin 54° - 4.9t^2)\mathbf{j}$<br>$\approx (16.458t)\mathbf{i} + (4.25 + 22.652t - 4.9t^2)\mathbf{j}$ |
| **Specific behaviours** |
| ✓ integrates correctly twice<br>✓ correct expression for $\mathbf{v}(t)$<br>✓ correct expression for $\mathbf{r}(t)$ |

(b) Determine

(i) the time that the projectile takes to travel between the pole and the wall. (2 marks)

| **Solution** |
|---|
| $4.25 + 22.652t - 4.9t^2 = 1.29$<br><br>$t = 4.75$ s |
| **Specific behaviours** |
| ✓ equates $\mathbf{j}$ coefficient of displacement to height<br>✓ solves for time |

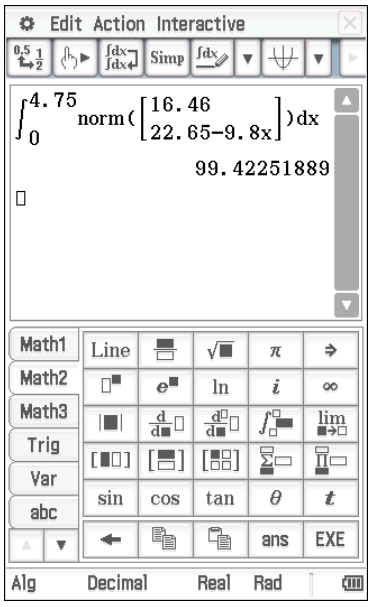
**See next page**

(ii) the speed of the projectile at the instant it hits the wall. (2 marks)

| **Solution** |
|---|
| $\mathbf{v}(4.75) = 16.46\mathbf{i} - 23.90\mathbf{j}$<br><br>$\lvert\mathbf{v}(4.75)\rvert = 29.0 \text{ ms}^{-1}$ |
| **Specific behaviours** |
| ✓ indicates velocity at correct time<br>✓ correct speed |

(iii) the length of the path taken by the projectile between the pole and the wall. (2 marks)

| **Solution** |
|---|
| $l = \int_0^{4.75} \lvert\mathbf{v}(t)\rvert \, dt$<br><br>$= 99.4 \text{ m}$ |
| **Specific behaviours** |
| ✓ writes integral for total distance<br>✓ correct distance |

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**Question 17** **(8 marks)**

Sphere $S$ has diameter $PQ$, where $P$ and $Q$ have coordinates $(6, -2, -3)$ and $(-2, 4, 1)$ respectively.

(a) Determine the vector equation of the sphere. (3 marks)

| **Solution** |
|---|
| $\overrightarrow{OC} = \frac{1}{2}(P+Q) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$<br><br>$r = \left\| \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\| = \sqrt{29}$<br><br>$\left\| \mathbf{r} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\| = \sqrt{29}$ |
| **Specific behaviours** |
| ✓ indicates position of centre<br>✓ indicates radius<br>✓ correct vector equation |

(b) Show that the point $(5, 5, 2)$ lies outside the sphere. (2 marks)

| **Solution** |
|---|
| $\left\| \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\| = \sqrt{34}$<br><br>Since $\sqrt{34} > \sqrt{29}$, point lies outside sphere. |
| **Specific behaviours** |
| ✓ calculates distance<br>✓ explains result |

(c) Show that the line with equation $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$ is tangential to the sphere. (3 marks)

| **Solution** |
|---|
| $\left\| \begin{pmatrix} 5+5\lambda \\ 1 \\ 6+2\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\| = \sqrt{29}$<br><br>$(3+5\lambda)^2 + (0)^2 + (7+2\lambda)^2 = 29 \Rightarrow \lambda = -1$<br><br>As $\lambda$ has a unique value, the line only intersects sphere at one point and so it must be a tangent. |
| **Specific behaviours** |
| ✓ substitutes line equation into sphere equation<br>✓ solves for $\lambda$<br>✓ explains result |

**See next page**

**Question 18** **(9 marks)**

Let $f(x) = \sqrt{x-2}$, $g(x) = \dfrac{6}{x}$ and $h(x) = f \circ g(x)$.

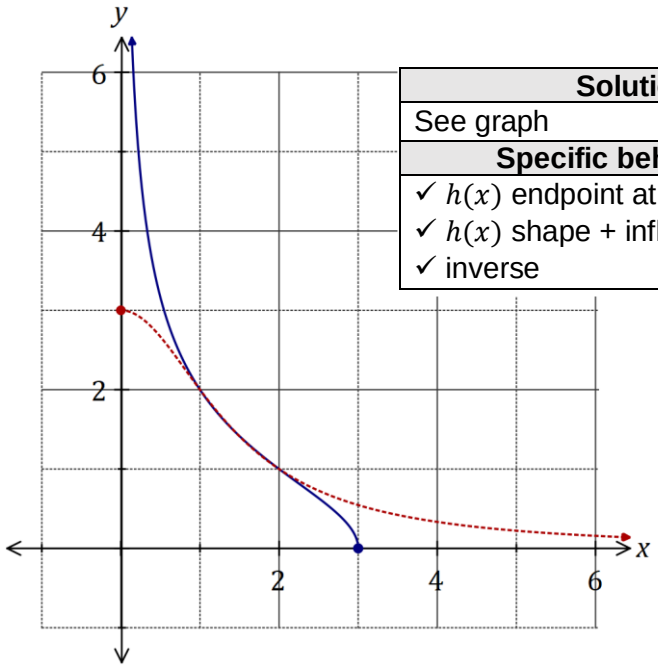
(a) Determine an expression for $h(x)$ and show that the domain of $h(x)$ is $0 < x \leq 3$. (3 marks)

| **Solution** |
|---|
| $h(x) = \sqrt{\dfrac{6}{x} - 2}$<br>$D_h$: (i) require $x > 0$ so that $\frac{6}{x} - 2 > 0$ and (ii) $\frac{6}{x} \geq 2 \Rightarrow x \leq 3$<br>Hence $D_h: \{x \in \mathbb{R}: 0 < x \leq 3\}$ |
| **Specific behaviours** |
| ✓ $h(x)$<br>✓ explains why $x > 0$<br>✓ explains why $x \leq 3$ |

(b) Determine an expression for $h^{-1}(x)$, the inverse of $h(x)$. (1 mark)

| **Solution** |
|---|
| $h^{-1}(x) = \dfrac{6}{x^2 + 2}$ (CAS) |
| **Specific behaviours** |
| ✓ correct expression |

(c) Sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$ on the axes below. (3 marks)




| **Solution** |
|---|
| See graph |
| **Specific behaviours** |
| ✓ $h(x)$ endpoint at $(3,0)$<br>✓ $h(x)$ shape + inflection $x \approx 2$<br>✓ inverse |

(d) Solve $h(x) = h^{-1}(x)$, correct to $0.01$ where necessary. (2 mark)

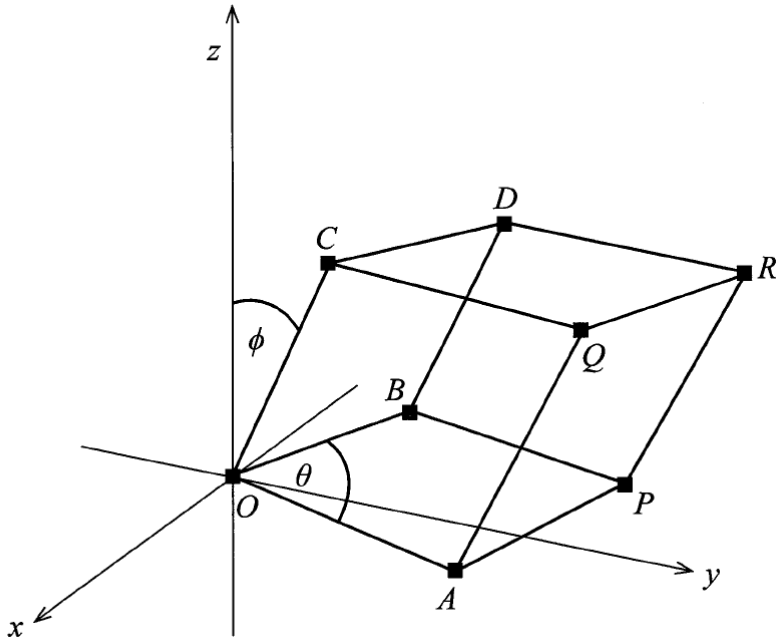
| **Solution** |
|---|
| $x = 1, \quad x = 2, \quad x \approx 1.46$ (CAS) |
| **Specific behaviours** |
| ✓ 2 correct solutions<br>✓ 3 correct solutions |

**Question 19** **(6 marks)**

A parallelepiped is a prism with each face a parallelogram. Let $OAPB$ be the parallelogram formed by the horizontal sides $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ where

$\underset{\sim}{a} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -8 \\ 2 \\ 0 \end{pmatrix}$.

The third side that forms the parallelepiped is $\underset{\sim}{c} = \overrightarrow{OC}$ where $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.




Let $\theta$ = the size of $\angle AOB$

$\phi$ = the angle between $\overrightarrow{OC}$ and the positive *z*-axis

(a) Determine $\underset{\sim}{a} \times \underset{\sim}{b}$. (2 marks)

| **Solution** |
|---|
| $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \times \begin{pmatrix} -8 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 54 \end{pmatrix}$ |
| **Specific behaviours** |
| ✓ determines the correct form of each component<br>✓ determines the components correctly |

**See next page**

The volume of any prism can be found by the formula *Volume = Area (base)* $\times$ *h,* where $h$ is the perpendicular height of the prism.

It is also true that $\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|=\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\sin\theta$.

(b) Explain why $\underset{\sim}{c}\bullet\left(\underset{\sim}{a}\times\underset{\sim}{b}\right)$ will determine the volume of the parallelepiped. (2 marks)

| **Solution** |
|---|
| $height = OC\cos\phi = \left\|\underset{\sim}{c}\right\|\cos\phi$<br>$Area\ base = \left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|$<br>$Vol = base\times height$<br>$= \left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|\left\|\underset{\sim}{c}\right\|\cos\phi = \left\|\underset{\sim}{c}\right\|\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|\cos\phi = \underset{\sim}{c}\bullet\left(\underset{\sim}{a}\times\underset{\sim}{b}\right)$ |
| **Specific behaviours** |
| ✓ justifies area of the parallelogram (base) is $\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|$<br>✓ justifies the perpendicular height is $\left\|\underset{\sim}{c}\right\|\cos\phi$ |

(c) Hence determine the exact volume of the parallelepiped. (2 marks)

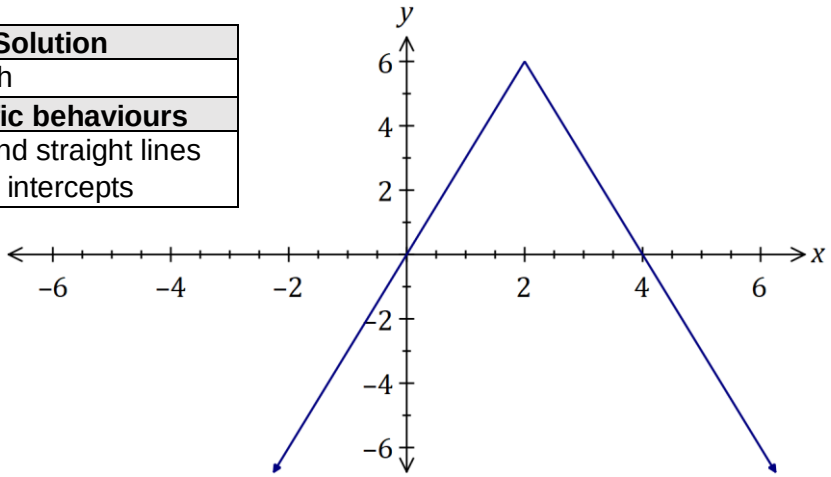
| **Solution** |
|---|
| $V = \underset{\sim}{c}\bullet\left(\underset{\sim}{a}\times\underset{\sim}{b}\right) = \begin{pmatrix}-1\\2\\5\end{pmatrix}\bullet\begin{pmatrix}0\\0\\54\end{pmatrix} = 0+0+270 = 270\,units^3$ |
| **Specific behaviours** |
| ✓ uses the correct vectors for the dot product<br>✓ determines the correct solution |

**See next page**

**Question 20** **(8 marks)**

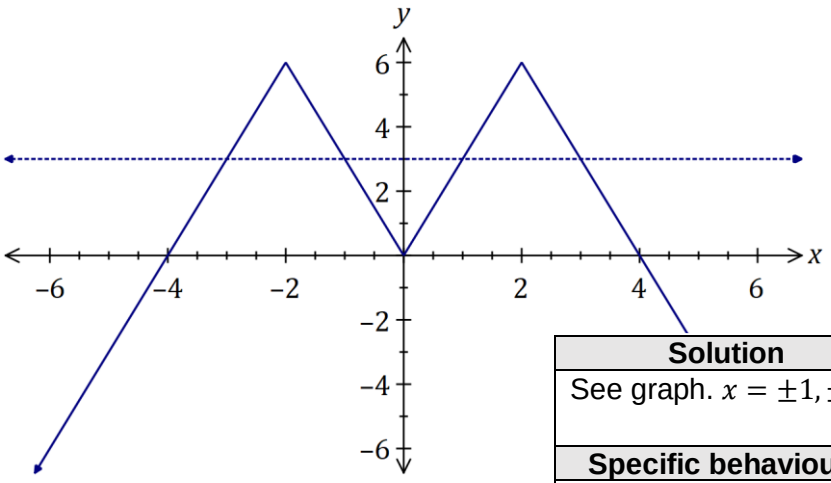
Let $f(x) = 6 - |3x - 6|$.

(a) Sketch the graph of $y = f(x)$ on the axes below. (2 marks)



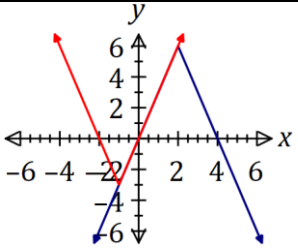

| Solution |
|---|
| See graph |
| **Specific behaviours** |
| ✓ cusp and straight lines |
| ✓ correct intercepts |

(b) Sketch the graph of $y = f(|x|)$ and hence solve $f(|x|) - 3 = 0$. (3 marks)




| Solution |
|---|
| See graph. $x = \pm 1, \pm 3$ |
| **Specific behaviours** |
| ✓ required sketch |
| ✓ adds $y = 3$ |
| ✓ solutions |

(c) The equation $f(x) = a|x + b| + c$ is true only for $-1 \leq x \leq 2$. Determine the value of each of the constants $a, b$ and $c$. (3 marks)

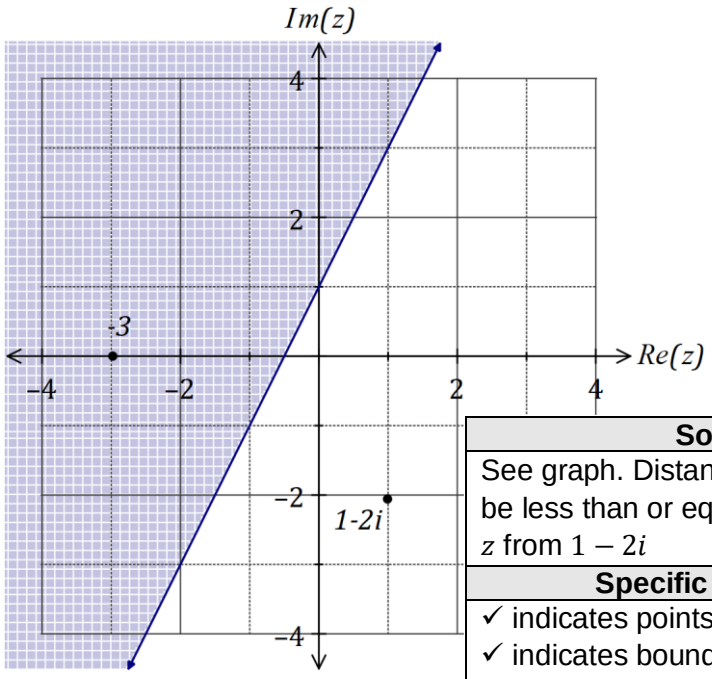
| Solution |
|---|
|  |
| $a = 3,\ b = 1,\ c = -3$ |
| **Specific behaviours** |
| ✓ sketch indicating situation |
| ✓ two correct values |
| ✓ all correct values |

**End of questions**

**Question 21** **(6 marks)**

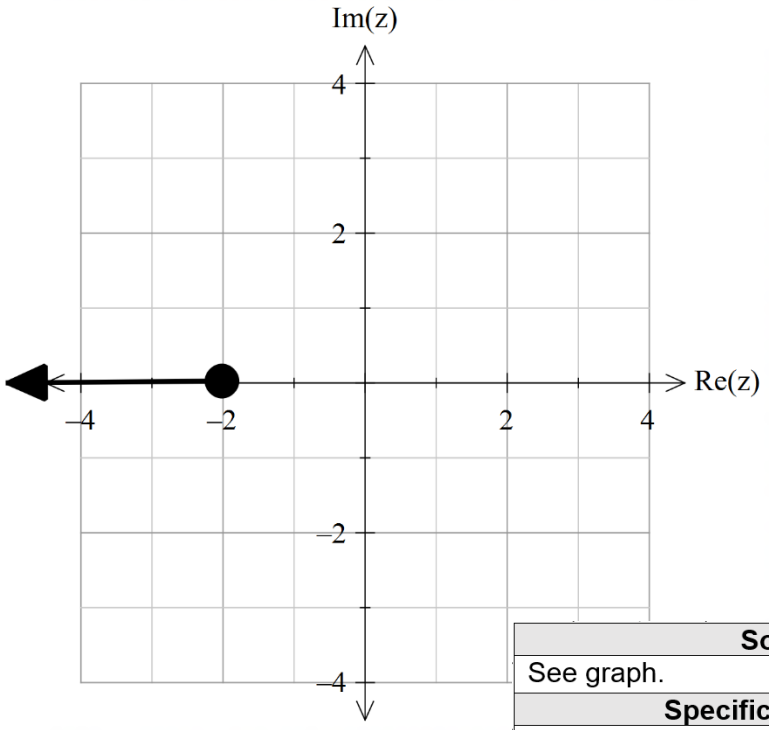
Sketch the locus of the complex number $z$ given by

(a) $|z+3| \leq |z-1+2i|$. (3 marks)




| Solution |
|---|
| See graph. Distance of $z$ from $-3$ must be less than or equal to the distance of $z$ from $1-2i$ |
| **Specific behaviours** |
| ✓ indicates points<br>✓ indicates boundary<br>✓ indicates correct half plane |

(b) $|z+2| = |z| - 2$. (3 marks)




| Solution |
|---|
| See graph. |
| **Specific behaviours** |
| ✓ indicates that $\mathrm{Re}\, z \leq -2$<br>✓ indicates that $\mathrm{Im}\, z = 0$<br>✓ indicates correct ray that includes origin |

**See next page**

See next page

Supplementary page

Question number: _____